't Hooft GR 13.2

$$ds^2 = -A\,dt^2 + B\,dr^2 + r^2[d\theta^2 + \sin^2\theta\, d\varphi^2]$$

$$E_r = E_1 = E(r), \quad E_\theta = E_\varphi = 0,$$

$$B_r = B_1 = B(r), \quad B_\theta = B_\varphi = 0.$$

$$J^\mu = 0.$$

$$F_{01} = \partial_0 A_1 - \partial_1 A_0 = E_1 = -F_{10}$$

$$F_{23} = \partial_2 A_3 - \partial_3 A_2 = B_1 = -F_{32}$$

$$\Rightarrow F^{01} = g^{0\mu} g^{1\nu} F_{\mu\nu} = g^{00} g^{11} F_{01}$$

$$= \frac{-1}{A}\frac{1}{B} E_1 = \frac{-E_1}{AB} = -F^{10}$$

$$\Rightarrow F^{23} = g^{2\mu} g^{3\nu} F_{\mu\nu} = \frac{1}{r^2}\frac{1}{r^2\sin^2\theta} F_{23}$$

$$= \frac{B_1}{r^4\sin^2\theta} = -F^{32}$$

$$\sqrt{-g} = \sqrt{AB}\, r^2 \sin\theta$$

(10.13) tells us $\partial_\mu(\sqrt{-g}\,F^{\mu\nu}) = -\sqrt{-g}\,J^\nu = 0$

we thus have

$$\cancel{\partial_1 \left[\sqrt{AB}\, r^2 \sin\theta \; F^{1\nu} \right]}$$

$$\partial_\mu \left[\sqrt{-g}\, F^{\mu\nu} \right] = 0.$$

Let $\nu = 0$, $\partial_1 \left[\sqrt{-g}\, F^{10} \right] = 0.$

Let $\nu = 1$, $\partial_0 \left[\sqrt{-g}\, F^{01} \right] = 0$

Let $\nu = 3$, $\partial_2 \left[\sqrt{-g}\, F^{23} \right] = 0$

$\nu = 2$, $\partial_3 \left[\sqrt{-g}\, F^{32} \right] = 0$

$$\Rightarrow \partial_0 \left[\sqrt{AB}\, r^2 \sin\theta \, \frac{-E_1}{AB} \right] = 0$$

$$\partial_0 \left[\frac{r^2 \sin\theta\, E_1}{\sqrt{AB}} \right] = 0$$

$$\partial_1 \left[\sqrt{AB}\, r^2 \sin\theta \, \frac{E_1}{AB} \right] = 0$$

$$\partial_1 \left[\frac{r^2 \sin\theta\, E_1}{\sqrt{AB}} \right] = 0.$$

$$\partial_2 \left[\sqrt{AB}\, r^2 \, \frac{\sin\theta\, B_1}{r^4 \sin^2\theta} \right] = 0.$$

$$\partial_2 \left[\frac{\sqrt{AB}}{r^2 \sin\theta} B_1 \right] = 0.$$

$$\partial_3 \left[\sqrt{AB}\, r^2 \, \frac{\sin\theta\, B_1}{r^4 \sin\theta} \right] = 0 \qquad \partial_3 \left[\frac{\sqrt{AB}}{r^2 \sin\theta} B_1 \right] = 0$$

$$T_{\mu\nu} = -F_{\mu\alpha} F_\nu{}^\alpha + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g_{\mu\nu}$$

$$F_{\alpha\beta} F^{\alpha\beta} = F_{10} F^{10} + F_{01} F^{01} + F_{23} F^{23} + F_{32} F^{32}$$

$$= 2\left[F_{10} F^{10} + F_{23} F^{23} \right]$$

$$= 2\left[(-E_1) \frac{E_1}{AB} + B_1 \frac{B_1}{r^4 \sin^2\theta} \right]$$

$$= 2\left[\frac{B_1^2}{r^4 \sin^2\theta} - \frac{E_1^2}{AB} \right]$$

$$-F_{0\alpha} F_0{}^\alpha = -F_{0\alpha} g^{\alpha\mu} F_{0\mu}$$

$$= -F_{01} g^{11} F_{01} = -E_1 \left(\frac{+1}{B}\right) E_1$$

$$= -\frac{E_1^2}{B}$$

$$\Rightarrow T_{00} = \frac{-E_1^2}{B} + \frac{1}{2}\left[\frac{B_1^2}{r^4 \sin^2\theta} - \frac{E_1^2}{AB} \right](-A)$$

$$= \frac{-E_1^2}{B} + \frac{1}{2}\frac{E_1^2}{B} - \frac{A B_1^2}{2r^4 \sin^2\theta}$$

$$= -\frac{1}{2}\frac{E_1^2}{B} - \frac{A B_1^2}{2r^4 \sin^2\theta}$$

$$T_{11} = -F_{1\alpha} F_1{}^{\alpha} + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g_{11}$$

$$= -F_{1\alpha}\, g^{\mu\alpha} F_{1\mu} + \frac{1}{2}\left[\frac{B_1^2}{r^4 \sin^2\theta} - \frac{E_1^2}{AB}\right] B$$

※ $$-F_{1\alpha}\, g^{\mu\alpha} F_{1\mu} = -F_{10}\, g^{00} F_{10}$$

$$= -E_1^2 \left(-\frac{1}{A}\right)$$

$$= \frac{E_1^2}{A}$$

$$\Rightarrow T_{11} = \frac{E_1^2}{A} + \frac{1}{2}\left[\frac{B_1^2}{r^4 \sin^2\theta} - \frac{E_1^2}{AB}\right] B$$

$$= \frac{E_1^2}{2A} + \frac{1}{2}\frac{B\, B_1^2}{r^4 \sin^2\theta}$$

$$-F_{2\alpha} F_2{}^{\alpha} = -F_{2\alpha}\, g^{\mu\alpha} F_{2\mu} = -F_{23}\, g^{33} F_{23}$$

$$= -\frac{B_1^2}{r^2 \sin^2\theta}$$

$$\Rightarrow T_{22} = \frac{-B_1^2}{r^2 \sin^2\theta} + \frac{1}{2}\left[\frac{B_1^2}{r^4 \sin^2\theta} - \frac{E_1^2}{AB}\right] r^2$$

$$= -\frac{B_1^2}{2r^2 \sin^2\theta} - \frac{E_1^2\, r^2}{2AB}$$

$$-F_{3\alpha} F_3{}^{\alpha} = -F_{3\alpha}\, g^{\mu\alpha} F_{3\mu}$$

$$= -F_{32}\, g^{22} F_{32}$$

$$= -\frac{B_1^2}{r^2}$$

$$T_{33} = -\frac{B_1^2}{r^2} + \frac{1}{2}\left[\frac{B_1^2}{r^4 \sin^2\theta} - \frac{E_1^2}{AB}\right] r^2 \sin^2\theta$$

$$= \frac{-B_1^2}{2r^2} - \frac{E_1^2\, r^2 \sin^2\theta}{2AB}$$

We still, have with $G_N = 1$,

$$R_{\mu\nu} = -8\pi\, T_{\mu\nu}.$$

$$R^{\sigma}_{\mu\lambda\nu} = \Gamma^{\sigma}_{\mu\nu,\lambda} - \Gamma^{\sigma}_{\mu\lambda,\nu} + \Gamma^{\sigma}_{\lambda\beta}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\sigma}_{\nu\beta}\Gamma^{\beta}_{\mu\lambda}$$

$$\Rightarrow R_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu} + \Gamma^{\lambda}_{\lambda\beta}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\lambda}_{\nu\beta}\Gamma^{\beta}_{\mu\lambda}$$

$$\cancel{R}\, T_{00} = -\frac{E_1^2}{2B} - \frac{A}{2r^4}\frac{B_1^2}{\sin^2\theta}$$

$$T_{11} = \frac{E_1^2}{2A} + \frac{B}{2r^4}\frac{B_1^2}{\sin^2\theta}$$

$$T_{22} = \frac{-B_1^2}{2r^2\sin^2\theta} - \frac{E_1^2 r^2}{2AB}$$

$$T_{33} = \frac{-B_1^2}{2r^2} - \frac{E_1^2}{2aB} r^2 \sin^2\theta$$

(11.15) → (11.26) applies as well to our case here, whereas we need to modify (11.27) :

$$R_{\mu\nu} = 0 \ (11.27) \Rightarrow R_{\mu\nu} = -8\pi T_{\mu\nu}.$$

⇒ (11.29) tells us

$$R_{00} = \frac{1}{2B}\left[A'' - \frac{A'B'}{2B} - \frac{A'^2}{2A} + \frac{2A'}{r}\right] = -8\pi T_{00}$$

we previously found $T_{00} = -\frac{1}{2}\frac{E_1^2}{B} - \frac{A}{2r^4}\frac{B_1^2}{\sin^2\theta}$

$$\partial_1\left[\frac{r^2\sin\theta E_1}{\sqrt{AB}}\right] = 0 , \qquad \partial_2\left[\frac{\sqrt{AB}}{r^2\sin\theta}B_1\right] = \partial_3\left[\frac{\sqrt{AB}}{r^2\sin\theta}B_1\right] = 0$$

$$\Rightarrow \frac{r^2 E_1}{\sqrt{AB}} = \frac{Q}{4\pi} \qquad \Rightarrow \frac{B_1\sqrt{AB}}{\sin\theta} \text{ rotationally invariant}$$

$$E_1 = \frac{Q\sqrt{AB}}{4\pi r^2} \qquad \Rightarrow \frac{B_1\sqrt{AB}}{\sin\theta} = \frac{S}{4\pi}$$

$$B_1 = \frac{S \sin\theta}{4\pi\sqrt{AB}}$$

S here is a constant independent of θ, ϕ.

$\Rightarrow T_{00} = -\frac{1}{2}\frac{E_1^2}{B} - \frac{A}{2r^4}\frac{B_1^2}{\sin^2\theta}$ gives

$$-\frac{1}{2\cancel{B}}\left[\frac{Q^2 A\cancel{B}}{16\pi^2 r^4}\right] - \frac{\cancel{A}}{2r^4\cancel{\sin^2\theta}}\frac{S^2\cancel{\sin^2\theta}}{16\pi\cancel{A}B}$$

$$= \frac{-1}{32\pi^2 r^4}\left[Q^2 A + \frac{S^2}{B}\right]$$

$\Rightarrow$ Einstein Eq becomes.

$$\frac{1}{\cancel{2B}}\left[A'' - \frac{\cancel{A'}B'}{2B} - \frac{A'^2}{2A} + \frac{2A'}{r}\right] = (\cancel{-8\pi})\frac{\cancel{1}}{2\cancel{4}\cancel{32}\pi^2 r^4}\left[BQ^2A + \frac{S^2}{\cancel{B}}\right]$$

$$A'' - \frac{A'B'}{2B} - \frac{A'^2}{2A} + \frac{2A'}{r} = \frac{1}{2\pi r^4}\left[Q^2AB + S^2\right]$$

$\Leftarrow$ for R_{00}, T_{00}.

* For T_{11}, R_{11}:

$$T_{11} = \frac{E_1^2}{2A} + \frac{B}{2r^4\sin^2\theta}B_1^2 = \frac{1}{2\cancel{A}}\left[\frac{Q^2\cancel{A}B}{16\pi^2 r^4}\right] + \frac{\cancel{B}}{2r^4\cancel{\sin^2\theta}}\left[\frac{S^2\cancel{\sin^2\theta}}{16\pi^2 A\cancel{B}}\right]$$

$$= \frac{1}{32\pi^2 r^4}\left[Q^2 B + \frac{S^2}{A}\right]$$

$$\Rightarrow \frac{1}{\cancel{2A}}\left[-A'' + \frac{A'B'}{2B} + \frac{A'^2}{2A} + \frac{2AB'}{rB}\right] = (-\cancel{8\pi})\frac{1}{2\cancel{4}\cancel{32}\pi^{\cancel{2}} r^4}\left[Q^2 B\cancel{A} + \frac{S^2}{\cancel{A}}\right]$$

$$-A'' + \frac{A'B'}{2B} + \frac{A'^2}{2A} + \frac{2AB'}{rB} = \frac{-1}{2\cancel{\pi} r^4}\left[Q^2AB + S^2\right]$$

Combining the T_{00} and T_{11} eq we still obtain.

$$\frac{2A'}{r} + \frac{2AB'}{rB} = 0$$

$$\frac{2}{rB}\left[A'B + AB'\right] = 0$$

$$\frac{2}{rB}\left[AB\right]' = 0$$

$$\Rightarrow \quad B = 1/A \qquad \text{still.}$$

*R_{22}, T_{22} Einstein Eq:

$$R_{22} = -\frac{\partial}{\partial\theta}\cot\theta - \left(\frac{r}{B}\right)' + \frac{2}{B} - \cot^2\theta - \frac{r}{B}\left(\frac{2}{r} + \frac{(AB)'}{2AB}\right) = -8\pi T_{22}$$

$$T_{22} = -\frac{1}{2r^2\sin^2\theta}B_1^{\,2} - \frac{r^2}{2AB}E_1^{\,2}$$

$$= \frac{-1}{2r^2\cancel{\sin^2\theta}}\,\frac{S^2\cancel{\sin^2\theta}}{16\pi^2\cancel{AB}} - \frac{\cancel{r^2}}{2\cancel{AB}}\,\frac{Q^2\cancel{AB}}{16\pi^2\cancel{r^4}\,r^2}$$

$$= \frac{-1}{32\pi^2 r^2}\left[S^2 + Q^2\right]$$

$$\Rightarrow \frac{-\partial}{\partial\theta}\cot\theta - \left(\frac{r}{B}\right)' + \cancel{\frac{2}{B}} - \cot^2\theta - \cancel{\frac{2}{B}} = \cancel{(8\pi)}\,\frac{\cancel{1}}{4\,\cancel{32\pi^2}r^2}\left[S^2 + Q^2\right]$$

$$-\frac{\partial}{\partial\theta}\cot\theta - \left(\frac{r}{B}\right)' - \cot^2\theta = \frac{1}{4\pi r^2}\left[S^2 + Q^2\right]$$

$$\cancel{\cot^2\theta} + 1 - \left(\frac{r}{B}\right)' - \cancel{\cot^2\theta} = \frac{1}{4\pi r^2}\left[S^2 + Q^2\right]$$

$$\Rightarrow \left(\frac{r}{B}\right)' = 1 - \frac{1}{4\pi r^2}\left[S^2 + Q^2\right]$$

Integrate with arbitrary constant $-2M$:

$$\frac{r}{B} = r + \frac{1}{4\pi r}\left[S^2 + Q^2\right] - 2M$$

$$\boxed{\frac{1}{B} = A = 1 + \frac{1}{4\pi r^2}\left[S^2 + Q^2\right] - \frac{2M}{r}}$$

It's clear that S here has the interpretation of the analogue of Q, the magnetic monopole charge.

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